Indian Statistical Institute, Bangalore

B.Math (Hons.) II Year, Second Semester

Mid-Sem Examination

Optimization

Time: 3 hours February 23, 2011 Instructor: Pl.Muthuramalingam

Maximum marks: 40

1. State Kuhn - Tucker theorem for convex sets and concave functions. [3]

2. Let J an *an* interval of R and $f : J \to R$ any concave function. Let $x_1 < x_2 < x_3$ be in J. Find an inequality connecting the difference quotients

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}, \quad \frac{f(x_3 \mid - f(x_1)}{x_3 - x_1}, \quad \frac{f(x_3) - f(x_2)}{x_3 - x_2},$$

   and prove it. [4]

3. Assume that $g : J \longrightarrow R$ where J is an interval of R satisfies

$$\frac{g(x_2) - g(x_1)}{x_2 - x_1} \geq \frac{g(x_3) - g(x_2)}{x_3 - x_2}$$

   for all $x_1 < x_2 < x_3$ in J. Show that g is a concave function. [3]

4. Let $1 \leq k < n$. Let $\mathbf{g} : R^n \longrightarrow R^k$ be a C^2 function ie $\mathbf{g} = (g_1, g_2, \cdots g_k)$ and $g_i : R^n \longrightarrow R$ is C^2 for each $i = 1, 2, \cdots, k$. Let $\mathbf{C} \in R^k$ be fixed . Let

$$S = \{\mathbf{x} \in R^n : \mathbf{g}(\mathbf{x}) = \mathbf{C}\}$$

   . Assume that $\mathbf{x}^* \in S$ and the square matrix $\left(\left(\frac{\partial g_i}{\partial x_j} (\mathbf{x}^*)\right)\right), i = 1, 2, \cdot k \quad ; \quad j = n, n-1, \cdots n-k+1$ is invertible so that for $\mathbf{x}^* = \left(x_1^* \cdots, x_k^*, x_{k+1}^* \cdots, x_n^*\right)$ there exists a C' function $\mathbf{h} : N \longrightarrow R^k$, $(x_1^*, \cdots x_{n-k}^*) \varepsilon N, N$ open with $\mathbf{h}(x_1^*, \cdots x_{n-k}^*) = (x_{n-k+1}^*, \cdots, x_n^*)$ and $\mathbf{g}(x_1, \cdots, x_{n-k}, \mathbf{h}(x_1, \cdots, x_{n-k})) = \mathbf{C}$, for all $(x_1, \cdots x_{n-k}) \in N$. Let $\mathbf{h} = (h_1, h_2, \cdots h_k)$. Show that each of the vectors $(e_j, \partial_{x_j}\mathbf{h}(\mathbf{a}))^t$ for $j = 1, 2, \cdots n-k$ is in kernel ($\equiv$ Null space) $\mathbf{g}'(\mathbf{x}^*)$. Here e_j is the row vector of length $n-k$ with 1 in the j th place and 0 elsewhere. Also $\mathbf{a} = (x_1^*, \cdots x_{n-k}^*)$. [5]

5. Let $V = A \bigoplus B$ where A, B, V are all finite dimensional linear spaces. If $a_1, a_2, \cdots a_r \varepsilon A$ are linearly independent and $b_1, b_2, \cdot b_r \varepsilon B$, then $\nu_1, \nu_2, \cdots \nu_r$, given by $\nu_j = (a_j, b_j)$ are linearly independent. [2]

6. (continuation of 4). Show that kernel $\mathbf{g}'(\mathbf{x}^*) =$

$$\{(a_1, a_2 \cdots, \cdot a_{n-k}, \sum_j a_j \partial_{x_j} h_1(\mathbf{a}), \cdots, \sum_j a_j \partial_{x_j} h_k(\mathbf{a}))^t$$

$a_i \varepsilon R$ for $i = 1, 2, \cdots n-k; j = 1, 2, \cdots n-k\}$. [5]

7. Let $g : R^n_+ \longrightarrow R$ be any continuous function and $y \varepsilon R$ is fixed. Let $\mathbf{w} = (w_1, w_2, \cdots, w_n)$ with $w_i > 0$ for each i. Show that min $\left\{\mathbf{w}.\mathbf{x} : \mathbf{x} \; \varepsilon \; R^n_+, g(\mathbf{x}) \geq y\right\}$ is attained. Here $R^n_+ = \{(x_1, x_2, \cdots, x_n) : x_i \geq 0$ for each $i\}$. [4]

8. Let $1 \leq k < n, \mathbf{h} : R^n \longrightarrow R^k$ be C' and $\mathbf{h} = (h_1, h_2 \cdots, h_k)$. Let $\mathbf{x}^* \in \mathbf{R}^n$. Assume that the $k \times k$ square matrix

$$\left(\left(\partial_{x_j} h_i(\mathbf{x}^*)\right)\right), i, j == 1, 2, \cdots k$$

is invertible, and $\mathbf{h}(\mathbf{x}^*) = \mathbf{0}$. Show that there exists $\sigma > 0$ and $\delta : (-\sigma, \sigma) \rightarrow R^n$ such that $(h_1(\delta(t)) - t, h_2(\delta(t)), \cdots h_k(\delta(t))) = \mathbf{0}$ for all t in $(-\sigma, \sigma)$ and $\delta(0) = \mathbf{x}^*$. [4]

9. State and prove Lagrange theorem. [3+4]

10. Let $f, h : R^2 \longrightarrow R$ be given by $f(x, y) = -(x^2 + y^2)$,

$h(x, y) = (x-1)^3 - y^2, \quad \mathbb{D} = \{(x, y) : h(x, y) \geq 0\}$

a) Show that $f(1, 0) = \max_{\mathbb{D}} f$. [3]

b) Show that we can not find $\lambda \geq 0$ such that

$$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial h}{\partial x}\right) at(1, 0) = 0$$

and

$$\left(\frac{\partial f}{\partial y} + \lambda \frac{\partial h}{\partial y}\right) at(1, 0) = 0.$$

[1]